The Callan–Symanzik equation

$$\mathcal{O} = \mathcal{O}\left(\lambda_a(\mu) \, , \, \ln E/\mu \right)$$

- $\langle \phi_1(x_1) \cdots \phi_n(x_n) \rangle \longrightarrow (2\pi)^4 \delta\left(\sum p_i\right) \; G(p_1, \dots, p_n)$

$$\sqrt{Z_i}\,\phi_i = (\phi_0)_i \quad \Longrightarrow \quad G_0(p_1, \dots, p_n) = \prod_i \sqrt{Z_i} \cdot G(p_1 \cdots p_n)$$

$$0 = \mu \frac{d}{d\mu} G_0 = \left(\mu \frac{d}{d\mu} \prod \sqrt{Z_i} \right) \cdot G + \left(\prod \sqrt{Z_i} \right) \mu \frac{d}{d\mu} G$$

$$0 = \left[\frac{1}{2} \sum_i \mu \frac{d}{d\mu} \ln Z_i + \mu \frac{d}{d\mu} \right] G$$

$$0 = \left[\sum_i \gamma_i + \mu \frac{d}{d\mu} \right] G$$

- $G = G(p_1, \dots p_n, \lambda_a(\mu), \mu)$

$\lambda_a \equiv$ all parameters
$\equiv$ all couplings &
all masses

$$\boxed{\left(\sum_i \gamma_i + \sum_a \beta_a \frac{\partial}{\partial \lambda_a} + \mu \frac{\partial}{\partial \mu} \right) G = 0}$$

- Ex
  - $\frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$ $\qquad \lambda_0 \longrightarrow m^2, \lambda$
  - $G \equiv G(p_1 \cdots p_n, \lambda(\mu), m^2(\mu), \mu)$

$$\left(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + \beta_{m^2} \frac{\partial}{\partial m^2} + n\gamma \right) G = 0$$

$$\underbrace{\left(\mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial \lambda} - \gamma_m m^2 \frac{\partial}{\partial m^2}\right.}_{} \left. + n\gamma \right) G = 0$$

$$\left(\mu \frac{d}{d\mu} + n\gamma \right) G = 0$$

- Formal solution

$$Z(\mu,\tilde{\mu}) = e^{-\int_{\tilde{\mu}}^{\mu} \frac{d\mu'}{\mu'}\gamma(\lambda(\tilde{\mu}))}$$

$$\mu\frac{d}{d\mu}Z(\mu,\tilde{\mu}) = -\gamma(\lambda(\mu))Z$$

$$\mu\frac{d}{d\mu}\left(Z^{-n}(\mu,\tilde{\mu})\,G(P,\lambda(\mu),m^2(\mu),\mu)\right) = 0$$

$$\Rightarrow Z^{-n}(\mu,\tilde{\mu})\,G(P,\lambda(\mu),m^2(\mu),\mu) = G(P,\lambda(\tilde{\mu}),m^2(\tilde{\mu}),\tilde{\mu})$$

$$\boxed{G(P,\lambda(\mu),m^2(\mu),\mu) = Z^{n}(\mu,\tilde{\mu})\,G(P,\lambda(\tilde{\mu}),m^2(\tilde{\mu}),\tilde{\mu})}$$

- Scattering amplitudes

$$\langle \phi_0 \, \phi_0 \rangle = \frac{Z_0}{p^2 - m_R^2} + \cdots$$

$$\downarrow \text{RG inv}$$

$$\downarrow$$

$$Z_0 = \text{RG inv}$$

$$\mu \frac{d}{d\mu} Z_0 = 0$$

$\Rightarrow$ LSZ

$$\mathcal{M}(p_1 \cdots p_n) = \prod_i^n \frac{p_i^2 - m_R^2}{\sqrt{Z_0}} \langle \hat{\phi}_0(p_1) \cdots \hat{\phi}_0(p_n) \rangle = \text{RG inv}$$

$$\boxed{\mu \frac{d}{d\mu} \mathcal{M}(p_1 \cdots p_n) = 0}$$

- "Single scale" observables

$$\mathcal{O} = \mathcal{O}(E, \lambda(\mu), m^2(\mu), \mu) \qquad [\mathcal{O}] = \Delta$$

$$\equiv E^{\Delta} F\left(\frac{E}{\mu}, \frac{m^2(\mu)}{E^2}, \lambda(\mu)\right)$$

$$= E^{\Delta} F\left(1, \frac{m^2(E)}{E^2}, \lambda(E)\right)$$

Ex $2 \to 2$

$$= E^{\Delta} \lambda(E) \left[P_0\left(\frac{m^2(E)}{E^2}\right) + \lambda(E) P_1\left(\frac{m^2(E)}{E^2}\right) + \dots \right]$$

- systematic resummation of $\lambda^n \left(\lambda \operatorname{Log} \frac{E}{\mu}\right)^m$ contribution
- does not automatically capture possible

"IR divergences" $\sim \ln \frac{m^2}{E^2}$

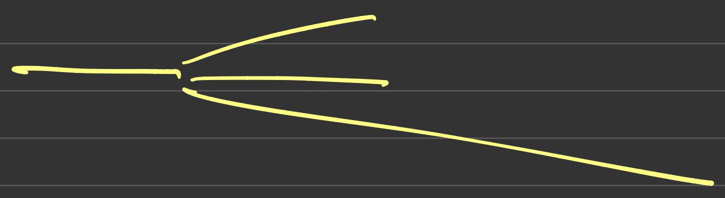
Ex

$- m^2 \ln m^2/E^2 \longrightarrow$

present in ϕ^4 but not in QED

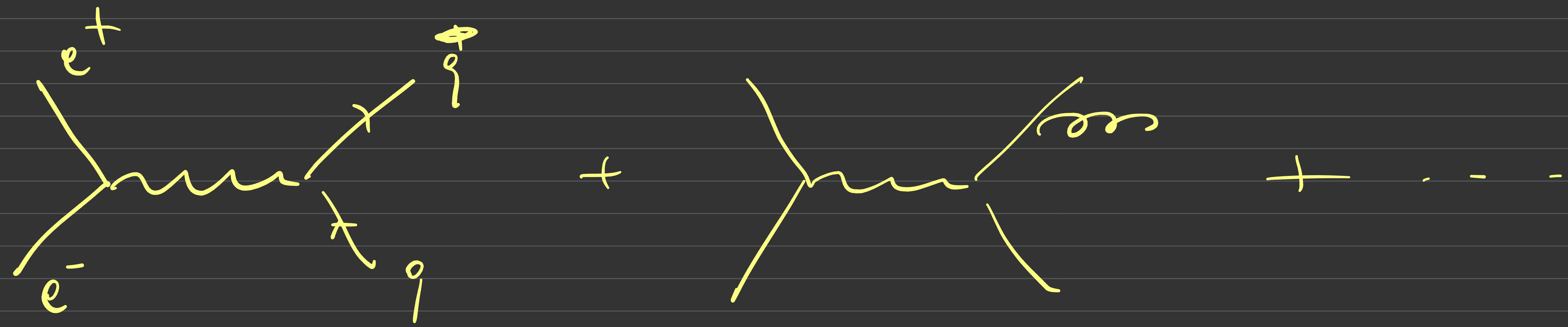
$- \lambda^4 \ln^2 \frac{m^2}{E^2} \longrightarrow$

collinear divergences often present



• Ex of pure single scale

$e^+e^- \longrightarrow$ hadrons at all orders in α_s



$$\sigma(s) = \frac{\alpha_E^2(s)}{s} \sum_n^\infty c_n \, \alpha_s^n(s)$$

Fixed points

$\lambda \equiv \lambda^*$ consistently at all μ's

- If
  - $m = 0$
  - $\lambda_a = \lambda_a^* \mid \beta_a(\lambda_a^*) = 0$

$\xrightarrow{\text{c.s.}}$

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_i n_i \gamma_i (\lambda^*)\right) G(p_1 \cdots p_n, \lambda^*, \mu)$$

- $Z_1(\mu, \tilde{\mu}) = e^{-\int_{\tilde{\mu}}^{\mu} \gamma(\lambda^*)\, d\ln\mu'} = \left(\frac{\tilde{\mu}}{\mu}\right)^{\gamma_*}$ $\quad \gamma_* \equiv \gamma(\lambda_*)$

$$\boxed{G\left(p,\ \lambda(\mu),\ \omega^2(\mu),\ \mu\right) = Z^{n}(\mu, \tilde{\mu})\, G\left(p,\ \lambda(\tilde{\mu}),\ \omega^2(\tilde{\mu}),\ \tilde{\mu}\right)}$$

• **Ex**

$$G_2(p, -p, \lambda^*, \mu) = Z^2(\mu, \tilde{\mu})\, G_2(p, -p, \lambda^*, \tilde{\mu})$$

$$\tilde{\mu} \equiv \sqrt{p^2} \qquad = Z^2(\mu, \sqrt{p^2})\, G_2(p, -p, \lambda^*, \sqrt{p^2})$$

$$= \left(\frac{\sqrt{p^2}}{\mu}\right)^{2\gamma_*} \frac{1}{p^2}\left(c_0 + c_1 \lambda^* + c_2 \lambda^{*2} \dots\right)$$

$$\propto p^{-2+2\gamma_*}$$

anomalus dimension

$$\Longrightarrow \langle \phi(x)\,\phi(0) \rangle \propto x^{-2-2\gamma_*}$$

as if $[\phi] = 1 + \gamma_*$ anomalous dimension